

Critical current of a superconducting wire via gauge/gravity duality

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We describe application of the gauge/gravity to study of depairing current in thin superconducting wires. The large number N of colors of the gauge theory is identified with the number of filled transverse channels in the wire. On the gravity side, the physics is described by a system of D3 and D5 branes intersecting over a line. We consider the ground state of the system at fixed electric current and find that there is a continuous phase transition at a critical current, in the universality class of the dissipative XY model. We discuss relation of our results to recent experiments on statistics of the switching current in nanowires.

Destruction of superconductivity by a current flowing through the superconductor has long been recognized as an important topic both from the viewpoint of fundamental physics and for applications (for a review, see Bardeen [1]). One expects that theoretical treatment should be the simplest for samples that are effectively one-dimensional (1d), i.e., wires in which the superconducting density depends on only one coordinate. Even for this case, however, a complete theoretical treatment of the transition at *fixed current* has not been forthcoming. The main difficulty lies, somewhat ironically, in the definition of a current-carrying *normal* state: the equilibrium Fermi distribution used by the conventional mean-field theory is clearly inadequate for the purpose as it carries no current at all.

A related but different problem is a transition caused not by a fixed current (sustained by an external battery) but by an *initial winding* of the order parameter. Experimentally, this is the condition appropriate for a thin superconducting ring. In this case, the supercurrent is only metastable: there are fluctuations that lower the free energy [2] (provided there is an amount of disorder or a finite temperature [3]); these have become known as phase slips. The relevant question to ask then is whether there is a maximal winding number density beyond which the metastability becomes classical instability.

To define a current-carrying normal state, one needs to include some mechanism that equilibrates the electrons with respect to momenta. One possible approach is to include weak scattering via a kinetic equation. Another, which we adopt here, is to start at the opposite extreme—a theory with strong electron-electron interactions. One may hope that for such a theory there is a complementary (dual) description in terms of weakly coupled collective modes. If these collective modes are the same as seen on the superconducting side, one will have a unified description applicable to both phases, which should make understanding the phase transition easier.

Recently, following the discovery of the gauge/gravity duality [4], carrying out this program has become practical. The gauge/gravity duality allows one to study a strongly coupled $SU(N)$ gauge theory with a large number N of colors by doing calculations in classical gravity,

albeit in a higher-dimensional spacetime. Here we use this method to study the depairing transition at fixed current. The number N of colors is taken to correspond to the number of populated transverse modes (channels) in the wire. Our conclusion is that at a critical current there is a continuous phase transition to the normal state in the universality class of the dissipative XY model. The duality can also be used to study transition due to an initial winding; we outline the strategy in what follows.

In the construction we use here, the $SU(N)$ gauge fields live in (3+1) dimensions, while the electrons only move in (1+1). The (3+1)-dimensional sector is conformal: qualitatively, this corresponds to an attractive $1/r$ potential between left- and right-moving electrons. Regardless of whether one considers this a realistic model of a 1d superconductor, the main result—a second-order transition at fixed current—may be universal enough to apply to experimental samples.

The large N suppresses interaction of the collective modes at the cutoff scale; in a wire, this corresponds to suppression of self-interaction of the pair field by the wire's cross-sectional area. As a result, in a given experiment, the near-critical behavior may be controlled by the ultraviolet Gaussian fixed point, rather than the stable infrared fixed point of the dissipative XY model. As we discuss at the end, this appears to be the case in recent experiments [5–7], at least at not too low temperatures.

We begin by establishing our convention for assembling electron operators into Dirac spinors. Assuming that superconductivity is due to a correlation between oppositely moving electrons in the same transverse mode, we expect it to show in correlation functions of the operator $\sum_{A=1}^N a_R^A b_L^A$, where the subscripts R and L designate the electron operators with positive and negative momenta, respectively. For this to be a color singlet, a_R should transform as N of the $SU(N)$ and b_L as \bar{N} (or vice versa). Thus, we define a 2-component Dirac spinor ψ as follows (omitting the color index A):

$$\psi = \begin{bmatrix} \sum_{k>0} (a_{Rk} e^{ikx} + b_{Rk}^\dagger e^{-ikx}) \\ \sum_{k<0} (a_{Lk} e^{ikx} + b_{Lk}^\dagger e^{-ikx}) \end{bmatrix}. \quad (1)$$

This is in the representation where the Dirac γ matrices are given by $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, and $\gamma^5 = \sigma_3$, in terms

of the Pauli matrices σ . The identification of the superconducting channel as $a_R b_L$ implies that a_{Rk}^\dagger creates a $k > 0$ electron, and b_{Lk}^\dagger a $k < 0$ electron (i.e., a_{Lk}^\dagger creates a $k < 0$ hole). With this convention, the upper and lower components of ψ have opposite electric charges, the superconducting channel is $\bar{\psi}\psi$ (where $\bar{\psi} = \psi^\dagger \gamma^0$), and an external electromagnetic potential couples to the axial current $\bar{\psi}\gamma^\mu\gamma^5\psi$.

To reproduce the physics of such a superconductor on the gravity side, we consider the system of N coincident D3 branes and a single D5 brane intersecting over a line. As common in applications of the gauge/gravity duality, in the large N , large 't Hooft coupling limit the D3 branes are replaced by their classical geometry while the D5 is considered as a probe, i.e., its effect on the geometry is neglected. The resulting geometry is that of a throat, of coordinate length R , pulled by the D3s out of the flat 10-dimensional (10d) spacetime:

$$ds^2 = \frac{1}{\sqrt{f}} (-dt^2 + d\mathbf{x}^2) + \sqrt{f} (d\Delta^2 + \Delta^2 d\phi^2) + \sqrt{f} (d\rho^2 + \rho^2 d\Omega_3^2), \quad (2)$$

where

$$f = 1 + \frac{R^4}{(\Delta^2 + \rho^2)^2}, \quad (3)$$

t and $\mathbf{x} = (x^1, x^2, x^3)$ are coordinates on the D3 worldvolume, Δ and ϕ are polar coordinates in the (x^8, x^9) plane, $\rho^2 = (x^4)^2 + \dots + (x^7)^2$, and $d\Omega_3^2$ is metric on a unit 3-sphere. In what follows, we use dimensionless units in which $R = 1$. The D3s are located at $\Delta = \rho = 0$, where the metric (2) has a degenerate horizon. This metric is suitable for calculations at zero temperature, the only case for which we present detailed calculations here. The results, however, do have a bearing on behavior at finite T , as we discuss towards the end.

Matter in the fundamental representation of $SU(N)$ (in our case, the electrons) is described by strings stretching between the D3s and a probe brane [8]. We consider the case when the probe D5 wraps x^1, ρ and the 3-sphere. Since $x \equiv x^1$ is the only spatial direction shared by the D5 and D3s, this brane intersection describes a theory of electrons that live on a (1+1)-dimensional defect but interact via a (3+1)-dimensional non-abelian gauge field. Overall the setup is similar to the system of D3 and D7 branes intersecting over a plane, in which the electrons are confined to move in *two* spatial dimensions [9, 10]. Note that in our case there are two directions, x^8 and x^9 , orthogonal to all branes. In complex notation, the displacement of the D5 relative to the D3s in the (x^8, x^9) plane is $\Delta e^{i\phi}$ and forms an order parameter suitable for description of superconductivity. The minimal distance between the D5 and D3s is the quasiparticle gap (in string units).

For a general D5 embedding,

$$\Delta = \Delta(t, x, \rho, \alpha_i), \quad (4)$$

$$\phi = \phi(t, x, \rho, \alpha_i), \quad (5)$$

where α_i are angles on the 3-sphere. Vibrations of the brane correspond to collective modes of the electron fluid. In the 't Hooft limit ($N \rightarrow \infty$ at fixed $\lambda = g_s N$, where g_s is the closed string coupling), quantum fluctuations of the brane are suppressed by the large value of the brane tension, so to the leading order the brane can be considered as a classical object. One can then explore various embedding ansatzes, which will typically have less coordinate dependence than the most general form (4)–(5). We assume throughout that $x^2 = x^3 = 0$. In addition, all embeddings we consider here are independent of α_i . With this restriction, the Dirac-Born-Infeld (DBI) action of the D5 brane is

$$S_{\text{DBI}} = -2\pi^2 T_5 \int dt dx d\rho \rho^3 f^{3/4} [-\det(G_{ab} + F_{ab})]^{1/2}, \quad (6)$$

where T_5 is the brane tension, G_{ab} for $a, b = t, x, \rho$ are the components of the induced metric, and $F_{ab} = \partial_a A_b - \partial_b A_a$ is a $U(1)$ gauge field (distinct from the usual electromagnetic field) on the D5 worldvolume. The classical dynamics of the brane is described by the Euler-Lagrange (EL) equations following from the total of (6) and a Wess-Zumino term that describes the coupling of the D5 to the background Ramond-Ramond field [11].

A spatially uniform winding of the order parameter can be described by (static) embeddings for which $\phi(x) = qx$, where q is a constant, while Δ depends on ρ only. The constant $\partial_x \phi$ sources worldvolume electric field $F_{t\rho}$ through the Wess-Zumino coupling. This leads to a system of coupled equations for Δ and $F_{t\rho}$. We do not consider this case further here and move on to description of a transition at fixed current.

We begin with static x -independent embeddings $\Delta = \Delta(\rho)$, $\phi = 0$, $A_t = A_t(\rho)$ with all other components of A_a equal to zero. As explained in [12] (in the context of a different D-brane system), the value $\mu = A_t(\infty)$ is the chemical potential for the density $\psi^\dagger \psi$. Turning to (1), we see that the $k > 0$ and $k < 0$ electrons are oppositely charged with respect to μ . Thus, in the superconductor, μ is conjugate to the electric current. The lines of the radial field $F_{t\rho}$ have nowhere on the brane to end, so the D5 must extend behind the horizon [12]. Hence the boundary condition $\Delta(\rho = 0) = 0$: at any nonzero current the superconductor is gapless.

The action (6) for this type of embedding is $S_{\text{DBI}} = -\int dt \mathcal{F}$, where \mathcal{F} is the free energy:

$$\mathcal{F} = 2\pi^2 T_5 \int dx d\rho \rho^3 \sqrt{f} (1 + \Delta_{,\rho}^2 - F_{t\rho}^2)^{1/2}. \quad (7)$$

We use notation $\Delta_{,\rho} \equiv \partial_\rho \Delta$. The Wess-Zumino term vanishes. The equation of motion for A_t shows that the

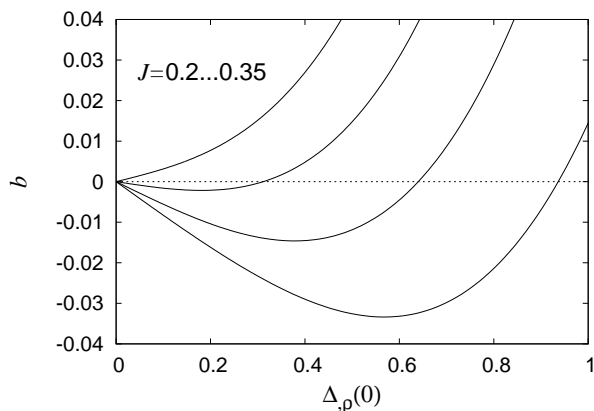


FIG. 1: Equation-of-state curves for several values of the current J . b is the asymptotic value of the D5 embedding, and $\Delta_{,\rho}(0)$ is its slope at the horizon. The current increases in increments of 0.05 from bottom to top.

current

$$J = \frac{1}{2\pi^2 T_5} \frac{\delta \mathcal{F}}{\delta F_{t\rho}} \quad (8)$$

is ρ -independent. Following the usual procedure of Legendre transforming to go from fixed μ to fixed J , we obtain the Legendre transformed free energy

$$\hat{\mathcal{F}} = 2\pi^2 T_5 \int dx d\rho (\rho^6 f + J^2)^{1/2} (1 + \Delta_{,\rho}^2)^{1/2}. \quad (9)$$

For each value of J , solutions to the corresponding EL equation form a one-parametric family with $\Delta_{,\rho}(0)$ as a parameter. The general behavior of a solution at large ρ is a constant: $\Delta(\infty) = b$.

We refer to the dependence of b on $\Delta_{,\rho}(0)$ as an equation-of-state (EOS) curve. These curves provide a convenient way to study the phase transition. Some representative ones are shown in Fig. 1. A nonzero asymptotic value $b \neq 0$ corresponds to an explicit breaking of the symmetry that shifts ϕ by a constant. Since we are interested in solutions that break this symmetry spontaneously, rather than explicitly, we look for nontrivial solutions with $b = 0$. In Fig. 1, such a solution corresponds to a second zero (if any) of the EOS curve; the first zero, at $\Delta_{,\rho}(0) = 0$, is the trivial solution. We see that second zeroes exist at smaller currents but as J increases they decrease in magnitude and eventually merge with the trivial solution and disappear. This is the behavior characteristic of a continuous phase transition. Numerically, the critical value of J is $J_c = 0.3197$.

To study fluctuations near the critical point, we consider the linearized EL equations with time and space dependence; these can be obtained by Legendre transforming the quadratic part of the DBI action (6). Upon

substitution $\Delta(t, x, \rho) = e^{-i\omega t + ikx} \Delta(\rho)$, the linearized equation for Δ reads

$$\frac{1}{\sqrt{C}} \partial_\rho (\sqrt{C} \Delta_{,\rho}) + \omega^2 f_0 \Delta - \frac{k^2 \rho^6 f_0^2}{C} \Delta + \frac{2\Delta}{C} = 0, \quad (10)$$

where $C = \rho^6 f_0 + J^2$ and $f_0 = 1 + 1/\rho^4$. At $\rho \rightarrow \infty$, this reduces to the spherical wave equation in (3+1) dimensions. At $\rho \rightarrow 0$, and $\omega \neq 0$, the leading asymptotics are $\Delta(\rho) \sim \rho e^{\pm i\omega/\rho}$. We choose the positive sign, corresponding to waves falling into the horizon. This choice will be the source of irreversibility (dissipation).

We consider real k and complex ω . Unstable modes of the trivial $\Delta \equiv 0$ embedding are eigenmodes of (10) with $\text{Im } \omega > 0$. They decay exponentially at both $\rho \rightarrow 0$ and $\rho \rightarrow \infty$. For these boundary conditions, ω^2 is purely real; hence ω is purely imaginary. For the moment, let us concentrate on the spatially uniform case $k^2 = 0$. By numerically solving (10) with Dirichlet boundary conditions, we find that, when $J < J_c$ but close to it, there is only one unstable mode, and its profile closely matches that of the nontrivial static solution to the nonlinear problem (9). This is evidence that the instability of the trivial embedding develops into one of the nontrivial embeddings we considered earlier. At $J = J_c$, the frequency crosses zero in a smooth, analytic manner, approximately as $\text{Im } \omega = -0.9(J - J_c)$, and the instability disappears. Analytically continuing $\text{Im } \omega$ to $J > J_c$, where the trivial embedding is stable, we interpret $|\text{Im } \omega|$ as a rate at which a small displacement of the D5 brane relaxes back to $\Delta \equiv 0$, i.e., a superconducting fluctuation decays into normal electrons.

Scaling of the relaxation rate with the control parameter near a critical point is described by a dynamical critical exponent, usually called y . The linear dependence of the rate on $J - J_c$ found above corresponds to $y = 1$. This, of course, is not the true infrared scaling. The latter is determined by interaction among the modes of (10): however weak initially, this interaction may renormalize to large strengths at large distances. The classical approximation to the D5's dynamics (corresponding to the leading large N approximation on the gauge theory side) does not see this renormalization but it can be used to determine the strength of the interaction—and, importantly, its sign—at the cutoff scale. To do that, we first consider a static but x -dependent configuration

$$\Delta e^{i\phi}(x, \rho; J) = \int dk e^{ikx} \Psi(k) \Delta^{(0)}(\rho; k, J), \quad (11)$$

where Ψ is a complex amplitude, and $\Delta^{(0)}$ is the unstable mode of (10) at the corresponding values of the parameters ($J < J_c$). Substituting this into the DBI action (6) and expanding to the fourth order in Ψ (but only to the second for terms containing derivatives with respect to x), we obtain a Ginzburg-Landau-type free energy. With

a suitable normalization of Ψ , it reads

$$\mathcal{F}_{\text{GL}} = \pi T_5 \left[-\int dk \Omega_0(k, J) |\Psi(k)|^2 + c \int dx |\Psi(x)|^4 \right], \quad (12)$$

where Ω_0 is $\text{Im } \omega$ of the unstable mode, and $\Psi(x)$ is the Fourier transform of $\Psi(k)$. $\Omega_0 \approx a(J_c - J) - a'k^2$ for J near J_c and small k^2 , while c is approximately constant. The constants a , a' , and c are all positive.

The rate of instability growth (at $J < J_c$) or relaxation (upon continuation to $J > J_c$) can be encoded into an effective action by allowing a weak time dependence in Ψ , going over to the Euclidean time via $t = -i\tau$, and adding the dissipative term

$$S_{\text{dissip}} = \pi T_5 \int dxd\Omega |\Omega| |\Psi(x, \Omega)|^2, \quad (13)$$

where Ω is the Euclidean frequency. The full Euclidean effective action is $S_E = \int d\tau \mathcal{F}_{\text{GL}} + S_{\text{dissip}}$. This theory is known as the dissipative XY model. It has been considered in applications to 1d superconductors [13] but, as far as we know, not specifically in reference to transition at a critical current. The gravity dual in effect provides a microscopic derivation of this model for that particular case.

As seen from (12), (13), the large T_5 (in our dimensionless units, $T_5 \sim \sqrt{\lambda}N$) suppresses fluctuations of Ψ . Thus, the ultraviolet behavior of the system is controlled by the Gaussian fixed point $c = 0$. It is known that this fixed point is unstable under renormalization, and the theory flows to a nontrivial infrared fixed point (for a summary of its properties, see Ref. [13]). However, since the true critical behavior will require a large length scale to set in, even a small deviation of J from J_c may lock the system in the domain controlled by the Gaussian fixed point. That is the case when the renormalization, say, for $J < J_c$ is cut off at the finite (correlation length) scale $\xi \sim \xi_0(1 - J/J_c)^{-1/2}$, where ξ_0 is some microscopic length, while the quartic coupling remains small.

Experimentally, superconducting nanowires have been observed to switch to the normal state at currents below the estimated critical current [5–7]. These experiments have found large fluctuations in the switching current, which persist down to low temperatures and which have been interpreted [5–7] as a consequence of phase slips. This interpretation is consistent with our conclusion that the transition at $J = J_c$ is second-order, as it implies that the free energy barrier suppressing phase slips can be almost completely removed by bringing J close to J_c . This in turn implies that the transition at $J = J_c$ must be continuous or at most very weakly first-order.

The following estimate suggests that the near-critical behavior in these experiments is controlled by the unstable Gaussian fixed point, rather than the stable XY one, at least at temperatures where thermally activated phase slips are thought to be the main effect. The exponential factor in the rate of thermal activation is $\exp(-\delta\mathcal{F}/T)$, where $\delta\mathcal{F}$ is the free energy barrier and T is the temperature. Near $J = J_c$, $\delta\mathcal{F}$ scales as the product of the free energy density and the correlation length, i.e., as $(J_c - J)^{2-\alpha-\nu}$, in terms of the conventionally defined critical exponents. For the Gaussian point, $\alpha = 0$ and $\nu = \frac{1}{2}$, so $\delta\mathcal{F} \sim (J_c - J)^{3/2}$. Curiously, this is the same scaling as obtained for a Josephson junction [14]. It has been found to provide a good fit to the data in Ref. [6] and for the amorphous samples in Ref. [7] (for the crystalline samples [7], the 5/4 power law has been found to be a better fit). In contrast, for a fixed point obeying hyperscaling, $2 - \alpha - \nu = 0$ and $\delta\mathcal{F}$ scales to a constant.

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